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Dedicated to the memory of B. M. Levitan

On the Variational Representation of Solutions to Some Quasilinear Equations and Systems of Hyperbolic Type on the Basis of Potential Theory

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Abstract. We demonstrate a method that permits to obtain generalized solutions for some quasilinear equations and systems of hyperbolic type. The corresponding variational principle is constructed using the theory of equilibrium of a potential in an external field.

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1. INTRODUCTION

The aim of the present paper is to demonstrate a method that permits one to obtain generalized solutions for some quasilinear equations and systems of hyperbolic type. This method is a variational principle similar to the well-known Lax–Oleinik principle for the case of one equation [1, 2] and its extension to some nonstrictly hyperbolic systems [3]. However, the novelty of our approach is that the corresponding variational principle is constructed by methods completely different from those known earlier. Namely, we use the theory of equilibrium of a potential in an external field. This permits us to construct a variational representation of generalized solutions of a strictly hyperbolic system in nondivergence form.

We construct a variational representation for the quasilinear equation

$$(\ln B)_t + B_x = 0, \tag{1.1}$$

where $B = B(t,x)$, $(t,x) \in \mathbb{R}_t \times \mathbb{R}$, and $B(0,x) = B_0(x) \geqslant 0$ is a smooth monotone bounded function, as well as for the system of equations

$$\begin{cases} \alpha_t + \dfrac{\beta-\alpha}{4}\,\alpha_x = 0, \\ \ln(\beta-\alpha)_t - \left(\dfrac{\alpha+\beta}{4}\right)_x = 0, \end{cases} \tag{1.2}$$

where $\alpha = \alpha(t,x)$ and $\beta = \beta(t,x)$, $(t,x) \in \mathbb{R}_t^+ \times \mathbb{R}_x^+$, with the initial conditions $\alpha(0,x) = \alpha_0(x)$ and $\beta(0,x) = \beta_0(x)$ and the boundary conditions $\alpha(t,0) = \beta(t,0)$.

Note that, in the smooth case, Eq. (1.1) is equivalent to the inviscid Burgers equation (Hopf equation)

$$B_t + BB_x = 0, \tag{1.3}$$

and system (1.2) is equivalent to the system

$$\begin{cases} \alpha_t + \dfrac{\beta-\alpha}{4}\,\alpha_x = 0, \\ \beta_t - \dfrac{\beta-\alpha}{4}\,\beta_x = 0. \end{cases} \tag{1.4}$$

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System (1.4) is strictly hyperbolic for $\alpha \neq \beta$, and hence its solutions admit discontinuities. System (1.4) is known as the *continuum limit of the Toda lattice*. Its "dispersive" regularization was studied by Deift and McLaughlin [5] (see also [6]). We are interested in discontinuous solutions of system (1.2) itself rather than its regularizations.

In the following, we show that the functions constructed according to our variational principle satisfy Eq. (1.1) or (1.3) and system (1.2) or (1.4) in the smoothness domain; if there is a discontinuity line, then Hugoniot relations associated with Eq. (1.1) or system (1.2) are valid.

System (1.2) is nondivergent, and hence we recall how one treats the Hugoniot relations in this case (see [4]).

Definition 1.1. Consider the system of equations

$$u_t + A(u)\, u_x = 0, \tag{1.5}$$

where $(t, x) \in \mathbb{R}_t \times \mathbb{R}$, $u = (u^1, u^2)$, and $A(u)$ is a 2×2 matrix. Suppose that a solution of this system can be locally represented in the form

$$u(t, x) = u_0 + H(x - Vt)(u_1 - u_0), \tag{1.6}$$

where $H(\theta)$ is the Heaviside function and V is the local direction of the discontinuity line. Then the Hugoniot relation associated with system (1.5) for the solution (1.6) is defined as the relation

$$-V(u_1 - u_0) + \int_0^1 A\big(\Phi(s; u_0, u_1)\big) \frac{\partial \Phi}{\partial s}(s; u_0, u_1)\, ds = 0, \tag{1.7}$$

where $\Phi(s; u_0, u_1)$ is some path joining u_0 and u_1 ($\Phi(0; u_0, u_1) = u_0$ and $\Phi(1; u_0, u_1) = u_1$).

Remark 1.1. Relation (1.7) substantially depends on the choice of Φ unless $A(u)$ is the total differential of some vector function $F(u)$, $DF(u) = A(u)$. In this case, system (1.5) is in divergence form, and relation (1.7) is independent on the choice of the path. For systems in nondivergence form, this ambiguity is intrinsic, and the choice of a specific relation (1.7) should be determined by a physical problem. This will be shown for system (1.2).

2. EQUILIBRIUM OF A FAMILY OF MEASURES OF VARIABLE MASS x IN AN EXTERNAL FIELD DEPENDING ON TIME t

To obtain a representation of solutions of problems (1.1)–(1.2), consider the following problem on the equilibrium of a potential in an external field. We define the potential corresponding to the logarithmic kernel, of a measure μ on $\mathbb{R}$ by the formula

$$V_\mu(\lambda) = \int \ln \frac{1}{|\lambda - z|}\, d\mu(z). \tag{2.1}$$

We normalize the measure to a parameter x:

$$\int d\mu(\lambda) = x.$$

Let $Q(\lambda)$ be a continuous function $Q : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ depending on the parameter t,

$$Q(t, \lambda) = Q_1(t, \lambda) + Q_0(\lambda), \qquad Q_1(0, \lambda) = 0. \tag{2.2}$$

This function will be called the external field. To ensure that the support S_x of the equilibrium measure $\overline{\mu}_x$ defined below is compact, we additionally require that

$$\lim_{|\lambda| \to +\infty} \frac{Q(\lambda)}{\ln |\lambda|} = +\infty.$$

Consider the variational problem

$$W_\mu := \frac{1}{2} \int V_\mu(\lambda)\, d\mu(x, \lambda) + \int Q(t, \lambda)\, d\mu(x, \lambda) \to \min_\mu \to W_{\overline{\mu}_x}. \tag{2.3}$$

Physically, problem (2.3) means that we seek a charge distribution $\overline{\mu}_x(\lambda, t)$ on the line so as to minimize the total energy of the charge in the external field (2.2). It is known [7–9] that there exists a unique minimizing measure $\overline{\mu}_x$, which is characterized by the equilibrium relation

$$V_{\overline{\mu}_x}(\lambda) + Q(t,\lambda) \begin{cases} = \gamma_Q & \text{on} \quad \operatorname{supp} \overline{\mu} =: S_x, \\ \geqslant \gamma_Q & \text{on} \quad \mathbb{R}; \end{cases} \tag{2.4}$$

here the constant γ_Q is called the *equilibrium constant*.

The family $\{S_x\}_{x>0}$ of supports of the equilibrium measures $\overline{\mu}_x$ is an important characteristic of the problem. It is a monotone family of compact sets (see [9]); i.e.,

$$S_{x'} \subseteq S_x, \qquad x' \leqslant x.$$

If the family $\{S_x\}_{x>0}$ is known, then one can reconstruct the input data of the problem, i.e., the external field Q, by the formula

$$Q(\lambda) = \int_0^\infty g_{S_x}(\lambda)\, dx, \tag{2.5}$$

where $g_K(\lambda)$ is the Green function of a compact set K. To find S_x, it is useful to consider the functional $F_Q(K)$ introduced in [10] on regular compact sets $K \subset \mathbb{R}$. Let ω_K be the Robin measure of a compact set K (i.e., ω_K is the equilibrium measure in problem (2.3)–(2.4) for $Q(\lambda) = 0$ and $\operatorname{supp} \mu \subset K$), and let $\operatorname{cap}(K)$ be the logarithmic capacity of K (i.e., $\operatorname{cap}(K) = \exp(-\gamma_0)$ in (2.4) for $Q(\lambda) = 0$ and $\operatorname{supp} \mu \subset K$). By definition,

$$F_Q(K) := -x \log \operatorname{cap}(K) + \int Q(\lambda)\, d\omega_K(\lambda). \tag{2.6}$$

It turns out (see [10, 8, 9]) that the support of the equilibrium measure S_x minimizes this functional; more precisely,

$$F_Q(K) \geqslant F_Q(S_x) = \gamma_Q \tag{2.7}$$

for each compact set K. By (Δ) we denote the class of external fields $Q(\lambda)$ such that the supports of the equilibrium measures S_x are intervals,

$$Q \in (\Delta) \Leftrightarrow S_x = \Delta_x := [\alpha(x), \beta(x)] \qquad \forall\, x > 0. \tag{2.8}$$

It is these initial data (Q_0 in (2.2)) in this class that are of interest to us in this paper. For example, it is known (see [8]) that each convex field Q belongs to (Δ); however, we are mainly interested in nonconvex Q_0 in (Δ).

3. THE INVISCID BURGERS EQUATION (HOPF EQUATION)

To obtain representations of solutions of problem (1.1), consider problem (2.3), (2.4) on the equilibrium of the potential in the external field $Q(t,\lambda)$. In this section, we consider fields Q even with respect to λ; hence we write $\operatorname{supp} \overline{\mu} = [-b(t,x), b(t,x)]$. Under the assumption that $Q \in (\Delta)$, the functional (2.6) has the form

$$-x \ln \frac{b}{2} + \frac{1}{\pi} \int_{-b}^{b} \frac{Q(t,\lambda)}{\sqrt{b^2 - \lambda^2}}\, d\lambda \to \min_b \qquad \text{for fixed } t \text{ and } x. \tag{3.1}$$

We take

$$Q(t,\lambda) = \lambda^2 t + Q_0(\lambda). \tag{3.2}$$

Then problem (3.1) becomes

$$-x \ln \frac{b}{2} + \frac{b^2}{2} t + \frac{1}{\pi} \int_{-b}^{b} \frac{Q_0(\lambda)}{\sqrt{b^2 - \lambda^2}}\, d\lambda \to \min_b \qquad \text{for fixed } t \text{ and } x. \tag{3.3}$$

It is known [9] that the field $Q(\lambda)$ and the family $\{[-b(x), b(x)]\}_{x>0}$ of supports of the equilibrium measures are related by the formula

$$Q(\lambda) = \int_0^\infty \ln \left| \frac{\lambda + \sqrt{\lambda^2 - b^2(x)}}{b(x)} \right| dx.$$

Theorem 3.1. *If the minimum in problem* (3.3) *is unique, then the function* $B(t, x) = b^2(t, x)$ *is a smooth solution of problem* (1.1); *the points* (t, x) *for which there are two minima locally form a curve along which a Hugoniot condition associated with* (1.1) *in the sense of Definition* 1.1 *holds.*

Proof. Since the minimum in (3.3) is attained at some point, it follows that the derivative with respect to b at that point is zero,

$$-\frac{x}{b} + bt + \frac{d}{db}\left(\frac{1}{\pi}\int_{-b}^{b} \frac{Q_0(\lambda)}{\sqrt{b^2-\lambda^2}}\, d\lambda\right) = 0. \tag{3.4}$$

However, it is known that, for $t = 0$, the minimization problem for the energy of the charge in the external field results in an initial measure with support $[-b_0(x), b_0(x)]$, $b_0^2(x) = B_0(x)$; i.e.,

$$\frac{d}{db}\left(\frac{1}{\pi}\int_{-b}^{b} \frac{Q_0(\lambda)}{\sqrt{b^2-\lambda^2}}\, d\lambda\right) = \frac{b_0^{-1}(b)}{b},$$

whence it follows that

$$\frac{1}{\pi}\int_{-b}^{b} \frac{Q_0(\lambda)}{\sqrt{b^2-\lambda^2}}\, d\lambda = \int_0^b \frac{b_0^{-1}(s)}{s}\, ds. \tag{3.5}$$

Thus from (3.4), we obtain

$$-\frac{x}{b} + bt + \frac{b_0^{-1}(b)}{b} = 0,$$

or

$$x = b^2 t + b_0^{-1}(b). \tag{3.6}$$

Since (3.6) is a characteristic relation for (1.1) for $B = b^2$, we see that we have proved the first assertion of Theorem 3.1.

If there are two points of global minimum, b_1 and b_2, in (3.3), then

$$-x \ln\frac{b_1}{2} + \frac{b_1^2}{2}t + \frac{1}{\pi}\int_{-b_1}^{b_1} \frac{Q_0(\lambda)}{\sqrt{b_1^2-\lambda^2}}\, d\lambda = -x \ln\frac{b_2}{2} + \frac{b_2^2}{2}t + \frac{1}{\pi}\int_{-b_2}^{b_2} \frac{Q_0(\lambda)}{\sqrt{b_2^2-\lambda^2}}\, d\lambda. \tag{3.7}$$

Relation (3.7) locally determines some curve $x(t)$. We differentiate (3.7) with respect to t and take into account (3.4) to obtain

$$-\dot{x} \ln\frac{b_1}{2} + \frac{b_1^2}{2} = -\dot{x} \ln\frac{b_2}{2} + \frac{b_2^2}{2},$$

or

$$-2\dot{x}(\ln b_2 - \ln b_1) + b_2^2 - b_1^2 = 0,$$

i.e.,

$$-\dot{x}(\ln b_2^2 - \ln b_1^2) + b_2^2 - b_1^2 = 0. \tag{3.8}$$

Relation (3.8) is obviously a Hugoniot relation for (1.1) in the sense of Definition 1.1. Thus we have proved the second assertion of Theorem 3.1.

Remark 3.1. In view of (3.5), the variational problem for the support of the measure has the form

$$-x \ln\frac{b}{2} + \frac{b^2}{2}t + \int_0^b \frac{b_0^{-1}(s)}{s}\, ds \to \min_b. \tag{3.9}$$

Note also that the classical variational principle for the inviscid Burgers equation $b_t + bb_x = 0$ is given by the extremum problem

$$\frac{(x-a)^2}{2t} + \int_0^a b_0(s)\, ds \to \min_a, \quad b = \frac{x - a_{\min}}{t} = b_0(a_{\min}), \tag{3.10}$$

or, if we write $\dfrac{x-a}{t} = b$,

$$\frac{b^2}{2}t + \int_0^{b_0^{-1}(b)} b_0(s)\, ds \to \min_b \quad \text{under the constraint} \quad x - bt - b_0^{-1}(b) = 0. \tag{3.11}$$

The solution of the conditional extremum problem (3.11) results in an analog of (3.9) for the inviscid Burgers equation. This is the relationship between our variational principles and traditional statements.

4. A HYPERBOLIC SYSTEM IN NONDIVERGENCE FORM

To obtain representations of solutions of problem (1.2), consider the problem on the equilibrium of the potential in the external field, similar to the problem considered in Section 3. Note that here the field is not necessarily an even function. Then the functional (2.6) has the form

$$-x\ln\frac{\beta-\alpha}{4}+\frac{1}{\pi}\int_\alpha^\beta\frac{Q(t,\lambda)}{\sqrt{(\lambda-\alpha)(\beta-\lambda)}}\,d\lambda\to\min_{\alpha,\beta} \tag{4.1}$$

for fixed t and x; here $\operatorname{supp}\overline{\mu}=[\alpha,\beta]$.

We take $Q(t,\lambda)=-\lambda t/2+Q_0(\lambda)$; then problem (4.1) becomes

$$-x\ln\frac{\beta-\alpha}{4}-t\frac{\alpha+\beta}{4}+\frac{1}{\pi}\int_\alpha^\beta\frac{Q_0(\lambda)}{\sqrt{(\lambda-\alpha)(\beta-\lambda)}}\,d\lambda\to\min_{\alpha,\beta} \tag{4.2}$$

for fixed t and x. Here the function Q_0 of the class (Δ) (see (2.8)) is related to the initial data by (2.5),

$$Q_0(\lambda)=\int_0^\infty g_{\Delta_x}(\lambda)\,dx,\quad\text{where}\quad g_{\Delta_x}(\lambda)=\ln\left|\frac{\lambda-a_0(x)+\sqrt{(\lambda-a_0(x))^2-4b_0(x)^2}}{2b_0(x)}\right|; \tag{4.3}$$

here

$$a_0(x):=\frac{\alpha_0+\beta_0}{2}(x),\qquad b_0(x):=\frac{\beta_0-\alpha_0}{4}(x).$$

Theorem 4.1. *If the minimum in problem* (4.2) *is unique, then the functions* $\alpha(t,x)$ *and* $\beta(t,x)$ *are a smooth solution of system* (1.2)*. The points* t,x *for which there are two such minima locally form a curve along which Hugoniot relations for* (1.2) *in the sense of Definition* 1.1 *hold.*

Proof. In the smooth case, system (1.2) is equivalent to system (1.4). By applying the hodograph transformation to (1.4), i.e., by introducing functions $t(\alpha,\beta)$ and $x(\alpha,\beta)$ such that

$$\begin{cases}\alpha\equiv\alpha\big(t(\alpha,\beta),\,x(\alpha,\beta)\big),\\ \beta\equiv\beta\big(t(\alpha,\beta),\,x(\alpha,\beta)\big),\end{cases} \tag{4.4}$$

we obtain the system

$$\begin{cases}\dfrac{\partial x}{\partial\alpha}+\dfrac{\beta-\alpha}{4}\dfrac{\partial t}{\partial\alpha}=0,\\ \dfrac{\partial x}{\partial\beta}-\dfrac{\beta-\alpha}{4}\dfrac{\partial t}{\partial\beta}=0\,.\end{cases} \tag{4.5}$$

We write $B(\alpha,\beta)\equiv\dfrac{1}{\pi}\displaystyle\int_\alpha^\beta\frac{Q_0(\lambda)}{\sqrt{(\lambda-\alpha)(\beta-\lambda)}}\,d\lambda$ and equate the partial derivatives of (4.2) with zero. Then we obtain the following relations for the functions $t(\alpha,\beta)$ and $x(\alpha,\beta)$:

$$\begin{cases}x-\dfrac{\beta-\alpha}{4}t=-(\beta-\alpha)\dfrac{\partial}{\partial\alpha}B(\alpha,\beta),\\ x+\dfrac{\beta-\alpha}{4}t=(\beta-\alpha)\dfrac{\partial}{\partial\beta}B(\alpha,\beta)\,.\end{cases} \tag{4.6}$$

Note that the function $B(\alpha,\beta)$ admits the representations

$$\begin{cases} (\beta-\alpha)B(\alpha,\beta) = \dfrac{2}{\pi}\displaystyle\int_\alpha^\beta Q_0(\lambda)(\beta-\lambda)\frac{d}{d\lambda}\sqrt{\frac{\lambda-\alpha}{\beta-\lambda}}\,d\lambda, \\ (\beta-\alpha)B(\alpha,\beta) = -\dfrac{2}{\pi}\displaystyle\int_\alpha^\beta Q_0(\lambda)(\lambda-\alpha)\frac{d}{d\lambda}\sqrt{\frac{\beta-\lambda}{\lambda-\alpha}}\,d\lambda\,. \end{cases} \tag{4.7}$$

Integrating by parts, we obtain

$$\begin{cases} (\beta-\alpha)B(\alpha,\beta) = -\dfrac{2}{\pi}\displaystyle\int_\alpha^\beta [Q_0'(\lambda)(\beta-\lambda)-Q_0(\lambda)]\sqrt{\frac{\lambda-\alpha}{\beta-\lambda}}\,d\lambda, \\ (\beta-\alpha)B(\alpha,\beta) = \dfrac{2}{\pi}\displaystyle\int_\alpha^\beta [Q_0'(\lambda)(\lambda-\alpha)+Q_0(\lambda)]\sqrt{\frac{\beta-\lambda}{\lambda-\alpha}}\,d\lambda\,. \end{cases}$$

By differentiating the first relation with respect to α and the second with respect to β, we obtain

$$\begin{cases} -B(\alpha,\beta)+(\beta-\alpha)\dfrac{\partial B(\alpha,\beta)}{\partial\alpha} = -\dfrac{2}{\pi}\displaystyle\int_\alpha^\beta \frac{Q_0'(\lambda)(\beta-\lambda)-Q_0(\lambda)}{-2\sqrt{(\lambda-\alpha)(\beta-\lambda)}}\,d\lambda, \\ B(\alpha,\beta)+(\beta-\alpha)\dfrac{\partial B(\alpha,\beta)}{\partial\beta} = \dfrac{2}{\pi}\displaystyle\int_\alpha^\beta \frac{Q_0'(\lambda)(\lambda-\alpha)+Q_0(\lambda)}{2\sqrt{(\lambda-\alpha)(\beta-\lambda)}}\,d\lambda\,, \end{cases}$$

or

$$\begin{cases} (\beta-\alpha)\dfrac{\partial B(\alpha,\beta)}{\partial\alpha} = \dfrac{1}{\pi}\displaystyle\int_\alpha^\beta Q_0'(\lambda)\sqrt{\frac{\beta-\lambda}{\lambda-\alpha}}\,d\lambda, \\ (\beta-\alpha)\dfrac{\partial B(\alpha,\beta)}{\partial\beta} = \dfrac{1}{\pi}\displaystyle\int_\alpha^\beta Q_0'(\lambda)\sqrt{\frac{\lambda-\alpha}{\beta-\lambda}}\,d\lambda\,. \end{cases} \tag{4.8}$$

By substituting (4.8) into (4.6), we obtain

$$\begin{cases} x-\dfrac{\beta-\alpha}{4}t = -\dfrac{1}{\pi}\displaystyle\int_\alpha^\beta Q_0'(\lambda)\sqrt{\frac{\beta-\lambda}{\lambda-\alpha}}\,d\lambda, \\ x+\dfrac{\beta-\alpha}{4}t = \dfrac{1}{\pi}\displaystyle\int_\alpha^\beta Q_0'(\lambda)\sqrt{\frac{\lambda-\alpha}{\beta-\lambda}}\,d\lambda\,. \end{cases} \tag{4.9}$$

By differentiating the first equation in (4.9) with respect to β and the second with respect to α, we find that

$$\begin{cases} \dfrac{\partial x}{\partial\beta}-\dfrac{\beta-\alpha}{4}\dfrac{\partial t}{\partial\beta}-\dfrac{t}{4} = -\dfrac{1}{\pi}\displaystyle\int_\alpha^\beta \frac{Q_0'(\lambda)\,d\lambda}{2\sqrt{(\lambda-\alpha)(\beta-\lambda)}}, \\ \dfrac{\partial x}{\partial\alpha}+\dfrac{\beta-\alpha}{4}\dfrac{\partial t}{\partial\alpha}-\dfrac{t}{4} = -\dfrac{1}{\pi}\displaystyle\int_\alpha^\beta \frac{Q_0'(\lambda)\,d\lambda}{2\sqrt{(\lambda-\alpha)(\beta-\lambda)}}\,. \end{cases} \tag{4.10}$$

We find t from (4.6):

$$\frac{t}{2} = \frac{\partial B(\alpha,\beta)}{\partial\alpha}+\frac{\partial B(\alpha,\beta)}{\partial\beta}\,.$$

Using (4.8), we further obtain

$$\frac{t}{2} = \frac{1}{\beta-\alpha}\cdot\frac{1}{\pi}\int_\alpha^\beta Q_0'(\lambda)\left(\sqrt{\frac{\beta-\lambda}{\lambda-\alpha}}+\sqrt{\frac{\lambda-\alpha}{\beta-\lambda}}\right)d\lambda = \frac{1}{\pi}\int_\alpha^\beta \frac{Q_0'(\lambda)\,d\lambda}{\sqrt{(\lambda-\alpha)(\beta-\lambda)}}\,.$$

By substituting this expression into (4.10), we readily arrive at (4.5). This proves the first assertion of Theorem 4.1.

If there are two points of global minimum (α_1, β_1) and (α_2, β_2) in (4.2), then

$$-x \ln \frac{\beta_1 - \alpha_1}{4} - t\frac{\alpha_1 + \beta_1}{4} + B(\alpha_1, \beta_1) = -x \ln \frac{\beta_2 - \alpha_2}{4} - t\frac{\alpha_2 + \beta_2}{4} + B(\alpha_2, \beta_2) \,. \tag{4.11}$$

Relation (4.11) locally determines some curve $x(t)$. We differentiate (4.11) with respect to t and take into account (4.6), thus obtaining

$$-\dot{x} \left[\ln \frac{\beta_2 - \alpha_2}{4} - \ln \frac{\beta_1 - \alpha_1}{4} \right] - \left[\frac{\alpha_2 + \beta_2}{4} - \frac{\alpha_1 + \beta_1}{4} \right] = 0 \,,$$

or

$$-\dot{x} \Big[\ln(\beta_2 - \alpha_2) - \ln(\beta_1 - \alpha_1) \Big] - \left[\frac{\alpha_2 + \beta_2}{4} - \frac{\alpha_1 + \beta_1}{4} \right] = 0 \,. \tag{4.12}$$

Now let us write out Definition 1.1 for system (1.2) in detail. To this end, we introduce the notation $\ln(\beta - \alpha) \equiv C$; then (1.2) becomes the system

$$\begin{cases} \alpha_t + \dfrac{e^C}{4}\, \alpha_x = 0, \\ C_t - \dfrac{e^C}{4} C_x - \dfrac{\alpha_x}{2} = 0 \,. \end{cases} \tag{4.13}$$

Following Definition 1.1, we obtain a relation on the discontinuity for system (4.13) and hence for (1.2):

$$-\dot{x} \left[\begin{pmatrix} \alpha_2 \\ C_2 \end{pmatrix} - \begin{pmatrix} \alpha_1 \\ C_1 \end{pmatrix} \right] + \int_0^1 \begin{pmatrix} e^{\Phi_2}/4 & 0 \\ -1/2 & -e^{\Phi_2}/4 \end{pmatrix} \begin{pmatrix} \partial\Phi_1/\partial s \\ \partial\Phi_2/\partial s \end{pmatrix} ds = 0 \,,$$

where $\begin{pmatrix} \Phi_1(s) \\ \Phi_2(s) \end{pmatrix}$ is the path in Definition 1.1 joining (α_1, C_1) with (α_2, C_2).

Next,

$$\begin{cases} -\dot{x}(\alpha_2 - \alpha_1) + \displaystyle\int_0^1 \frac{1}{4} e^{\Phi_2} \Phi_{1s}\, ds = 0, \\ -\dot{x}(C_2 - C_1) + \displaystyle\int_0^1 \left(-\frac{1}{2}\Phi_{1s} - \frac{1}{4} e^{\Phi_2} \Phi_{2s} \right) ds = 0 \end{cases}$$

and

$$\begin{cases} -\dot{x}(\alpha_2 - \alpha_1) + \displaystyle\int_0^1 \frac{1}{4} e^{\Phi_2} \Phi_{1s}\, ds = 0, \\ -\dot{x}(C_2 - C_1) - \dfrac{1}{2}(\alpha_2 - \alpha_1) - \dfrac{1}{4} \left(e^{C_2} - e^{C_1} \right) = 0 \,. \end{cases}$$

From this, recalling the definition of $C \equiv \ln(\beta - \alpha)$, we obtain

$$\begin{cases} -\dot{x}(\alpha_2 - \alpha_1) + \displaystyle\int_0^1 \frac{1}{4} e^{\Phi_2} \Phi_{1s}\, ds = 0, \\ -\dot{x}\big[\ln(\beta_2 - \alpha_2) - \ln(\beta_1 - \alpha_1)\big] - \dfrac{1}{2}(\alpha_2 - \alpha_1) - \dfrac{1}{4}\big[(\beta_2 - \alpha_2) - (\beta_1 - \alpha_1)\big] = 0 \,. \end{cases} \tag{4.14}$$

It is easily seen that the second equation in (4.14) coincides with (4.12) and the path (Φ_1, Φ_2) can always be chosen in such a way that the first equation in (4.14) will be satisfied as well. Thus we have proved the second assertion of Theorem 4.1.

Thus, starting from a representation based on a meaningful physical approach, we have selected "admissible" Hugoniot relations from the set of all Hugoniot relations possible in accordance with Definition 1.1 for a quasilinear hyperbolic system in nondivergence form. The presence of a variational representation can be viewed as an alternative form of the condition of entropy growth on a shock wave.

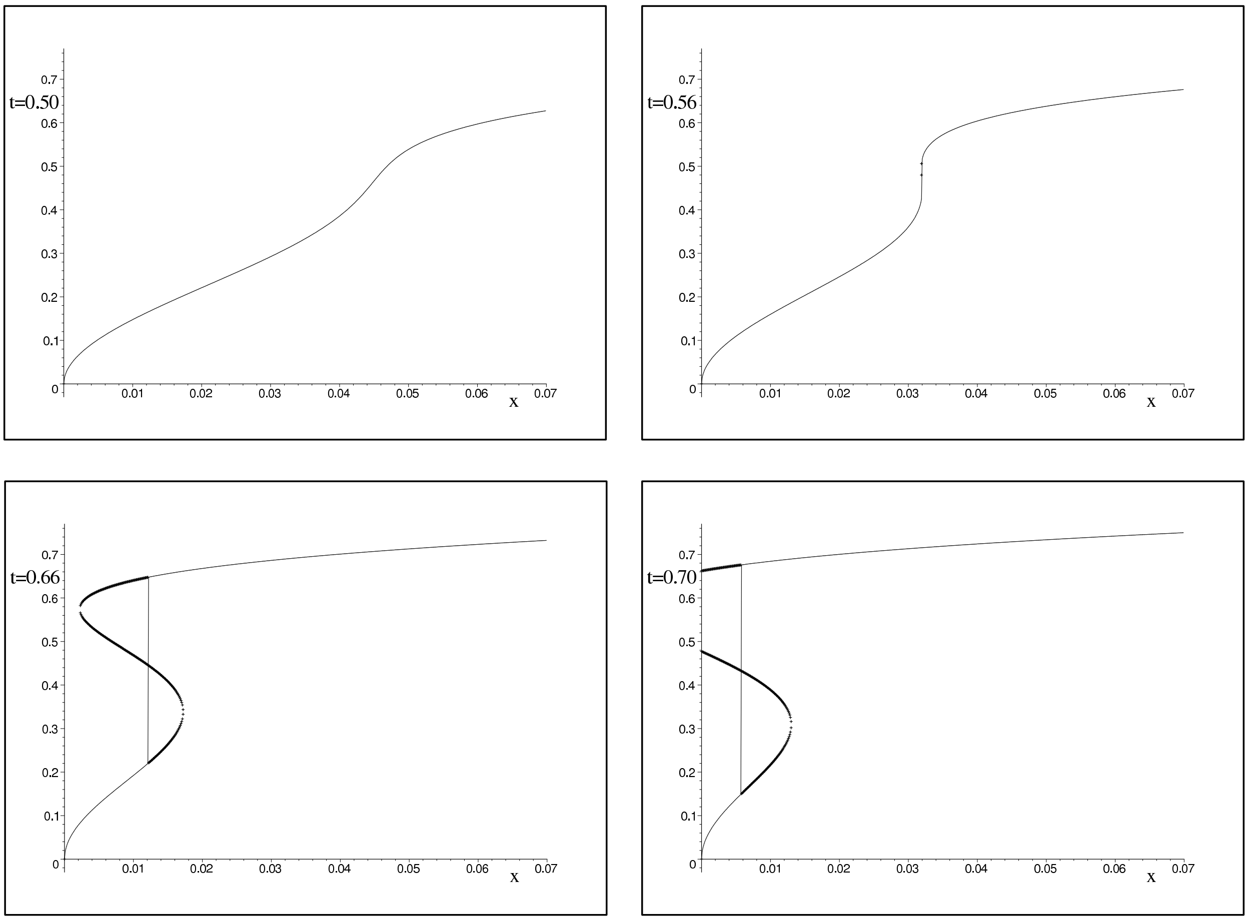

Fig. 1. Extrema of the functional (3.1) at various times t.

5. EXAMPLES

Figure 1 shows the graphs of the extrema $b(x)$ of the functional (3.1) at various times t. The vertical line (shock wave) joins the global minima. The initial conditions for Eq. (1.1) correspond to the initial external field

$$Q_0(\lambda) := 3\lambda^6 - 2\lambda^4 + \lambda^2.$$

Figure 2 shows the solutions of system (1.2) at various times t; these solutions are computed as the global minima of the functional (4.2) under the initial conditions (4.3) corresponding to the initial external field

$$Q_0(\lambda) := \lambda^4 - \frac{\lambda^2}{2} + 4\lambda.$$

Note that there is a shock wave that arises and propagates for $t > 6$.

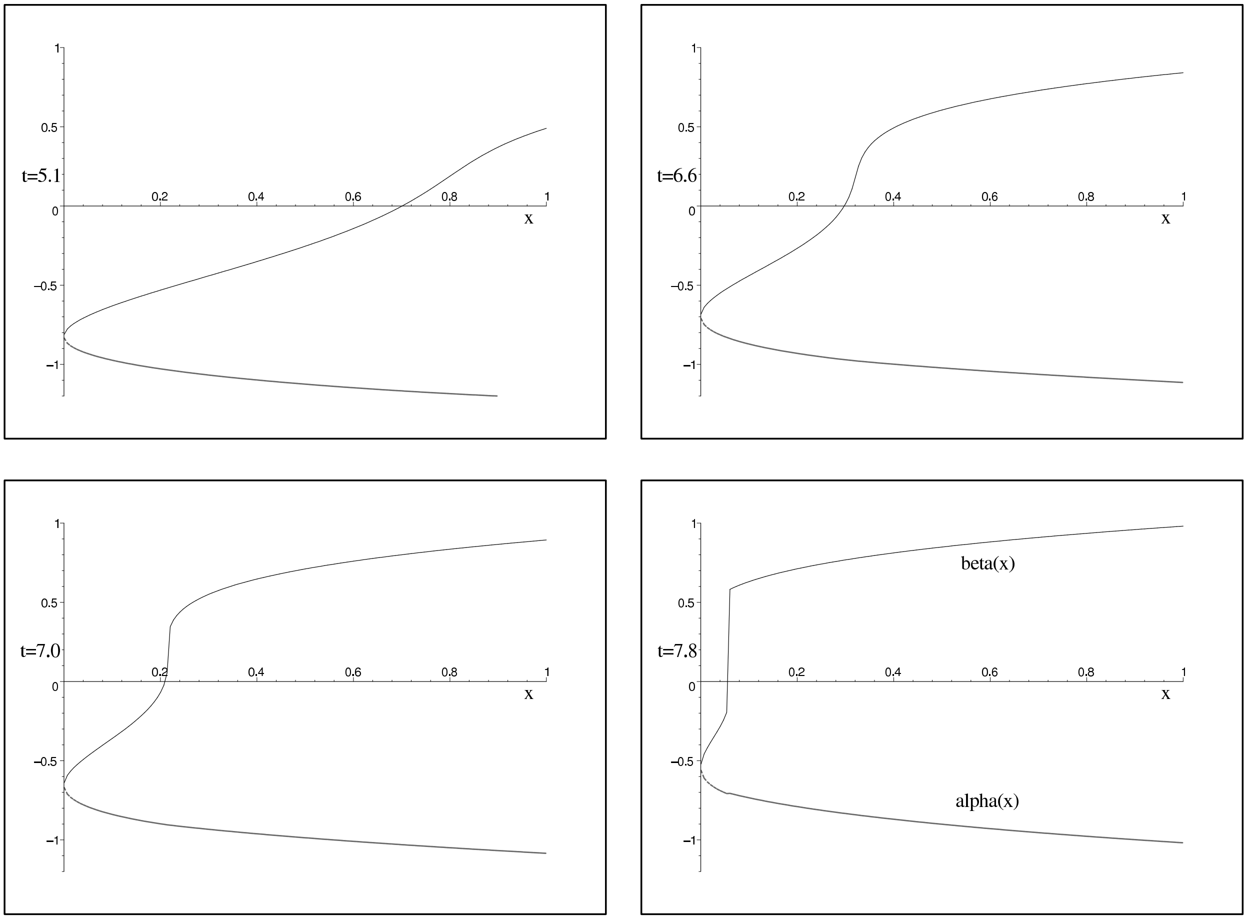

Fig. 2. The onset of a shock wave as a result of the solution of the variational problem.

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